GENERALIZATION OF RUNGE'S THEOREM TO APPROXIMATION BY ANALYTIC FUNCTIONS

BY

ANNETTE SINCLAIR

B.S., Central Missouri State College, 1940
A.M., University of Illinois, 1945

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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In this paper Runge's Theorem on approximation by a rational function on a closed set is generalized to approximation on a more general point set by a function analytic and nonvanishing except on certain preassigned points. The result obtained is applied in solving a problem proposed by Ketchum[1] on the construction of an analytic function of arbitrarily rapid growth.

Walsh[2] gives a generalized statement of Runge's Theorem in terms of a necessary and sufficient condition for approximation. He lets C be an arbitrary closed point set of the extended plane and assumes given points z_k (not necessarily denumerable) lying in only a finite number of regions into which C separates the plane. Then a necessary and sufficient condition that *every* function analytic on C can be uniformly approximated on C by a rational function whose poles lie in the points z_k is that at least one point z_k lie in each of the regions into which C separates the plane.

In the present paper the set S on which we are approximating may have an infinite number of components, provided no sequential limit point of the collection of components belongs to S. Such a set whose components are closed will be referred to as an *S-set*. On an S-set approximation by a rational function is, in general, not possible. We use as an approximating function a function which is analytic except on a preassigned C_S-set — a set which consists of the sequential limit points of the collection of components of S and of

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* The author is greatly indebted to Professor P. W. Ketchum, who directed the preparation of the thesis of which this is an abstract.

1. P. W. Ketchum, **On the Possible Rate of Growth of an Analytic Function**, Trans. A. M. S., vol. 49 (1941) pp. 211-228.
2. J. L. Walsh, **Interpolation and Approximation**, A. M. S. Colloq. Pub., vol. 20, New York, 1935.

precisely one point from each component of the complement of S which contains no sequential limit point. The approximation obtained is not only uniform on S but is such that the closeness of approximation can be preassigned independently for each component of S. If the restriction is placed on the function to be approximated that it be nonvanishing on S and that its logarithm be single-valued on S, the approximating function can be required to have not only its singularities but also its zeros on a preassigned C_S-set.

Theorem 1: Suppose that S is an S-set and that C is any C_S-set. Let $M(z)$ be any function which is nonvanishing and analytic on S and such that log $M(z)$ is single-valued on S. Let $\{\epsilon_i\}$ be any sequence of positive constants. Then there exists a function $f(z)$ nonvanishing and analytic except possibly on C such that, for every i, $|M(z) - f(z)| < \epsilon_i$ when z is any point of S_i (where S_i denotes a component of S).

This result is applied to the problem of the construction of an analytic function having arbitrarily rapid growth.

Ketchum has considered point sets such that, for any given function $G(z) \geqq 0$, there exists a function $f(z)$ which is analytic except possibly where $G(z)$ is unbounded and which satisfies the inequality $|f(z)| \geqq G(z)$ for every z of the given set. In the publication of these results he proposed a corresponding problem when the additional restriction is placed on the function $f(z)$ that it be nonvanishing except on certain specified points of the complement of the set. In particular, suppose S_1, S_2, - - - - - - - - - is an infinite sequence of simply connected regions whose closures are non-intersecting and whose only sequential limit point is the point at infinity. Then does there exist a nonvanishing integral function $f(z)$ such that $|f(z)| \geqq M_i$, for z in S_i, where $\{M_i\}$ is any preassigned sequence of positive constants?

In this paper the generalization of Runge's Theorem obtained is applied to prove that the above question can be answered in the affirmative. A corresponding result is obtained for a less restricted point set.

Before stating the theorem proved we give two more definitions. If S is an S-set, a set in the complement of S which contains the sequential limit points of S will be called an *E_S-set.* Let S be an S-set having an infinite nested sequence of components $\{S_i\}$ and let E be an E_S-set. Then $\{S_i\}$ will be said to be *E-free* if there exists N such that, for any $i > N$, no point of E belongs to the component of the complement of S connecting S_i and S_{i+1}.

Theorem 2: Let S be an S-set and let E be any E_S-set. Suppose that S has no E-free infinite nested sequence of components. Then, if $M(z)$ is any function which is bounded on each component of S, there exists a function $g(z)$ which is nonvanishing and analytic except possibly on E and which satisfies the inequality

$$|g(z)| > |M(z)|$$

when z is any point of S.

VITA

Annette Sinclair was born in Missouri August 14, 1916. She received the B. S. degree from Central Missouri State College with honorary mention in 1940 and the A. M. degree from the University of Illinois in 1945. She is a member of Phi Beta Kappa, Phi Kappa Phi, and Sigma Xi.

She taught seven years in the elementary and high schools of Carroll County, Missouri, and worked two years in the applied fields of statistics and actuarial science. While a graduate student, she was a teaching assistant in mathematics at the University of Illinois.

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